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LETTER TO THE EDITOR

An exact solution to the problem of a single-quantized field mode interacting with a certain multilevel atomic system

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Abstract. Exact solutions are given to the eigenvalue problem of a single, quantized field mode interacting in the dipole approximation with a single atom which has $N-1$ levels of one parity and one level of the opposite parity.

In this letter we consider an atomic system consisting of N levels, where $(N-1)$ levels are of one parity, and the other level is of the opposite parity, so that transitions can take place from the unique level to any of the other levels through the dipole coupling. The situation is depicted schematically in figure 1, where for convenience we have

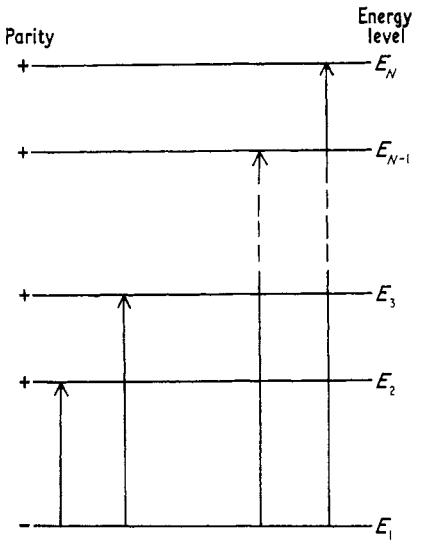

Figure 1. Transitions allowed in the dipole approximation between one state of negative parity and $N-1$ states of positive parity.

taken the unique level to have negative parity and the lowest energy. We assume that there is only one field mode, and we use a quantum-mechanical model. In addition to being more general, the quantum approach is basically simpler than the semi-classical in that the hamiltonian is time independent in the former case, and we therefore have to solve only the time-independent Schrödinger equation.

In two previous papers (Swain 1972, 1973a) we have given exact solutions to the simpler problem of a two-level atom interacting with a single quantized field mode. Our results were expressed in the form of continued fractions, which have been used before in the following references to state exact solutions of related problems. Autler and Townes (1955) and Swain (1973b) have considered the corresponding problem in which the quantized field is replaced by a classical field (the semi-classical problem). The connection between the quantum and semi-classical treatments has been discussed by Swain (1973c). Stenholm and Lamb (1969) and Shimoda (1970) have independently developed a density-matrix approach to the semi-classical problem where pumping and decay of the two atomic energy levels is taken into account. However, as far as the author is aware, the present letter is the first to apply the continued fraction approach to atomic systems having more than two levels.

Explicitly, our hamiltonian is

$$H = \sum_{k=1}^{N} E_k|k\rangle\langle k| + \omega a^\dagger a + \sum_{k=2}^{N} (g_{1,k}a + g_{1,k}^* a^\dagger)(|1\rangle\langle k| + |k\rangle\langle 1|) \tag{1}$$

where E_k denotes the energy of the kth atomic state, $|k\rangle$, ω the energy of the field mode, a and $a^\dagger$ the usual annihilation and creation operators for the field, and $g_{1,k}$ the coupling constant for transitions from state $|1\rangle$ to state $|k\rangle$. We have made the dipole approximation, but at no stage do we make the rotating wave approximation.

Our approach is to find the eigenvectors of H by solving the Schrödinger equation

$$H|\lambda\rangle = E_\lambda|\lambda\rangle. \tag{2}$$

Of course, once this problem is solved, we can express all the properties of the system, including the time-dependent ones, in terms of these eigensolutions. We expand $|\lambda\rangle$ in terms of the eigenstates of the non-interacting system, $|k, n\rangle$:

$$|\lambda\rangle = \sum_{n=0}^{\infty}\sum_{k=1}^{N} A_{k,n}|k, n\rangle, \tag{3}$$

where n is the photon occupation number, and the $A_{k,n}$ are coefficients to be determined. The latter will be functions of the eigenvalue E_λ, but we do not show this explicitly here. By substituting (3) into (2), and equating to zero separately each coefficient of $|k, n\rangle$, we obtain the following set of N difference equations:

$$(E_\lambda - E_1 - n\omega)A_{1,n} = \sum_{k=2}^{N} (g_{1,k}(n+1)^{1/2}A_{k,n+1} + g_{1,k}^* n^{1/2}A_{k,n-1}) \tag{4}$$

$$(E_\lambda - E_k - n\omega)A_{k,n} = g_{1,k}(n+1)^{1/2}A_{1,n+1} + g_{1,k}^* n^{1/2}A_{1,n-1} \qquad k = 2, 3, 4, \ldots, N. \tag{5}$$

By substituting from (5) into (4) we can obtain a difference equation in the $A_{1,n}$ only:

$$\left[E_\lambda - E_1 - n\omega - \sum_{k=2}^{N}\left(\frac{|g_{1,k}|^2(n+1)}{E_\lambda - E_k - (n+1)\omega} + \frac{|g_{1,k}|^2 n}{E_\lambda - E_k - (n-1)\omega}\right)\right]A_{1,n}$$
$$= \left(\sum_{k=2}^{N}\frac{g_{1,k}{}^2[(n+1)(n+2)]^{1/2}}{E_\lambda - E_k - (n+1)\omega}\right)A_{1,n+2} + \left(\sum_{k=2}^{N}\frac{(g_{1,k}^*)^2[n(n-1)]^{1/2}}{E_\lambda - E_k - (n-1)\omega}\right)A_{1,n-2}. \tag{6a}$$

This equation is of the form

$$D_n A_{1,n} = R_n A_{1,n+2} + R_{n-2}^* A_{1,n-2} \tag{6b}$$

and it has solutions of the form

$$A_{1,s+2p} = A_{1,s} \prod_{j=1}^{p} \frac{R^*_{s+2j-2}}{\alpha_{s+2j}} \qquad p = 1, 2, 3, \ldots \tag{7a}$$

$$A_{1,s-2p} = A_{1,s} \prod_{j=1}^{p} \frac{R_{s-2j}}{\beta_{s-2j}} \qquad p = 1, 2, 3, \ldots \tag{7b}$$

for any non-negative integer s. In (7*b*), only those values of p are allowed which ensure $s-2p \geqslant 0$. The quantities α_n and β_n have yet to be determined. $A_{1,s}$ is fixed (apart from a phase factor) by the condition that $|\lambda\rangle$ be normalized. We have chosen the particular solution (7) because it has the correct physical properties. (That is, as $|g_{1,k}| \to 0$ we recover the eigenstates of the non-interacting system. When this limit is taken, s corresponds to the number of photons present.)

If we assume first $n = s+2p$ in (6*b*) and substitute from (7*a*), and then assume $n = s-2p$ and substitute from (7*b*), we find the following equations for α_n and β_n:

$$\alpha_{s+2p} = D_{s+2p} - \frac{|R_{s+2p}|^2}{\alpha_{s+2p+2}} \qquad p = 1, 2, 3, \ldots \tag{8a}$$

$$\beta_{s-2p} = D_{s-2p} - \frac{|R_{s-2p-2}|^2}{\beta_{s-2p-2}} \qquad p = 1, 2, 3, \ldots . \tag{8b}$$

These equations show that the α_n and β_n may be expressed as continued fractions. We now assume $n = s$ in (6*b*) and substitute from (7*a*) and (7*b*), when we obtain a relation between the α and β. This relation may be written in a particularly neat form if we extend (8) to include $p = 0$ so that α_s and β_s may be defined. The relation may then be written

$$\alpha_{s+2}\beta_s = |R_s|^2. \tag{9}$$

α and β are defined by the relations (8) in terms of E_λ, which has not yet been determined. However, (9) gives a further condition of the α and β, and it may be regarded as the equation which determines E_λ. It therefore plays the role of an eigenvalue equation.

Once E_λ has been evaluated explicitly, the α and β, and thus the $A_{1,s\pm 2p}$, can be found. The $A_{k,s\pm 2p\pm 1}$ can be found from (5). Thus

$$A_{k,s\pm 2p\pm 1} = \gamma_{k,s\pm 2p\pm 1} A_{1,s\pm 2p}, \tag{10}$$

where

$$\gamma_{k,s+2p+1} = \frac{g_{1,k}(s+2p+2)^{1/2}R^*_{s+2p} + g^*_{1,k}(s+2p+1)^{1/2}\alpha_{s+2p}}{\alpha_{s+2p}[E_\lambda - E_k - (s+2p+1)\omega]} \tag{11a}$$

and

$$\gamma_{k,s-2p-1} = \frac{g_{1,k}(s-2p)^{1/2}\beta_{s-2p-2} + g^*_{1,k}(s-2p-1)^{1/2}R_{s-2p-2}}{\beta_{s-2p-2}[E_\lambda - E_k - (s-2p-1)\omega]}. \tag{11b}$$

Substituting the expressions (7) and (11) for the $A_{k,n}$ into (3), we finally obtain for the

eigenket $|\lambda\rangle$

$$|\lambda\rangle = A_{1\,s}(\lambda)\Bigg\{\sum_{p=0}^{s/2-1}\Bigg[\Bigg(|1,s-2p\rangle+\sum_{k=2}^{N}\gamma_{k,s-2p-1}(\lambda)|k,s-2p-1\rangle\Bigg)\prod_{j=1}^{p}\frac{R_{s-2j}(\lambda)}{\beta_{s-2j}(\lambda)}\Bigg]$$
$$+|1,s\rangle+\sum_{p=s/2+1}^{\infty}\Bigg[\Bigg(|1,s+2p\rangle+\sum_{k=2}^{N}\gamma_{k,s+2p+1}(\lambda)|k,s+2p+1\rangle\Bigg)$$
$$\times\prod_{j=1}^{p}\frac{R^{*}_{s+2j-2}(\lambda)}{\alpha_{s+2j}(\lambda)}\Bigg]\Bigg\}, \tag{12}$$

where for simplicity we have assumed s to be even and we have shown explicitly those quantities on the right-hand side of (12) which depend on E_λ.

Our results apply of course to the two-level system previously treated if we take $N = 2$, but our expressions take a different form to those presented in Swain (1973a). Although the expressions we obtain are complex, it is straightforward to approximate the continued fractions to obtain explicit expressions, or to evaluate them numerically, as may be verified by consulting the references previously quoted.

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